XIX. *The Coefficient of Viscosity of Air.*

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Communicated, with the addition of two Notes, by Professor G. G. STOKES, *P.R.S.*

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[PLATE 42.]

Origin and Purpose of the Investigation.

THREE years ago I entered on a series of researches relating to the internal friction of metals, little calculating, when I did so, that the task which I had set myself would occupy almost the whole of my spare time from that date to this. So, however, it has been, and one of the many causes of delay has been the necessity of making a re-determination of the coefficient of viscosity of air; for the resistance of the air played far too important a part in my investigations to permit of its being either neglected or even roughly estimated. The coefficient of viscosity of air may, according to MAXWELL, be best defined by considering a stratum of air between two parallel horizontal planes of indefinite extent, at a distance r from one another. Suppose the upper plane to be set in motion in a horizontal direction with a velocity of v centimetres per second, and to continue in motion till the air in the different parts of the stratum has taken up its final velocity, then the velocity of the air will increase uniformly as we pass from the lower plane to the upper. If the air in contact with the planes has the same velocity as the planes themselves, then the velocity will increase $\frac{v}{r}$ centimetres per second for every centimetre we ascend. The friction between any two contiguous strata of air will then be equal to that between either surface and the air in contact with it. Suppose that this friction is equal to a tangential force f on every square centimetre, then

$$f = \mu \frac{v}{r},$$

where μ is the coefficient of friction. If L, M, T represent the units of length, mass, and time, the dimensions of μ are $L^{-1}MT^{-1}$.

Several investigators have attempted to determine the coefficient of viscosity of air, and the following table shows how very widely the results obtained differ among each other :—

TABLE I.

Author.*	Coefficient of viscosity of air in C.G.S. units.	Temperature in degrees Centigrade.
G. G. STOKES, from BAILY'S pendulum experiments.	·000104	°
MEYER, from BESSEL'S experiments	·000275	
MEYER, from GIRAULT'S experiments	·000384	
MEYER	·000360	18
MEYER (second paper)†	·000333	8·3
" "	·000323	21·5
" "	·000366	34·4
MAXWELL	·000200	18

Further, MAXWELL finds the coefficient of viscosity of air to be independent of the pressure and to vary directly as the absolute temperature.‡ The above author gives the following formula for finding μ, the coefficient of viscosity, at any temperature θ° C.:—

$$\mu = \cdot 0001878(1 + \cdot 00365\theta).$$

MAXWELL offers an explanation of the difference existing between his own results and those of MEYER, but states that "he has not found any means of explaining the difference between his own results and those of Professor STOKES." Professor STOKES has, however, been good enough to inform me that, as at the time of making his deductions from BAILY'S experiments it was not known that the coefficient of viscosity of air was independent of the pressure, but, on the contrary, was assumed by him to vary directly as the pressure, the resistance offered by the residual air in BAILY'S partial vacua was underestimated, and, as a consequence, the deduced coefficient of viscosity was too small. It is to be hoped that Professor STOKES will at some future period apply the necessary corrections, but as this has not yet been done, and as we have still no explanation of the discrepancies existing between the other values of μ given in Table I., I wished to make some independent observations on the viscosity of air for the purpose of ascertaining how far these would agree with those of MAXWELL, in which I was inclined to place great confidence.

MAXWELL employed the method of torsional vibrations of disks placed each between two parallel fixed disks at a small, but easily measurable distance, in which case, when the period of vibration is long, the mathematical difficulties of determining the motion of the air are greatly diminished. This method appeared to be a very good one, but, as I wished to make my determinations under conditions similar to those

* For references see MAXWELL'S Bakerian Lecture, 'Phil. Trans.,' vol. 156, 1866, p. 249.

† MEYER has more recently made other determinations of the coefficient, for which see the end of the paper.

‡ This result does not seem to be confirmed by other experimenters. (See the end of the paper.)

which held in my experiments on the internal friction of metals, I have employed the torsional vibrations of cylinders or spheres attached to a horizontal cylindrical bar and moving in a sufficiently unconfined space. The mathematical difficulties connected with the use of vibrating spheres are not so serious, but those in which cylinders are concerned are very considerable. They both, however, have been surmounted by Professor G. G. STOKES in his valuable paper "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums,"* and to this paper I am indebted for the mathematics essential to the purpose of the present inquiry.

Description of Apparatus and Mode of Experimenting.

A wire, *a b* (Plate 42, fig. 1), was suspended in the axis of an air-chamber, W, made of two concentric copper cylinders enclosing between them a layer of water. The outer diameter of the air-chamber was 4 inches, the inner diameter 2 inches, and the length $4\frac{1}{2}$ feet. Resting on the top of the air-chamber and wedged into it was a stout T-shaped piece of brass, C, to the lower extremity of which was clamped one end of the wire. The lower extremity of the wire was soldered or clamped at *b* to a vertical cylindrical copper bar *b* Q, which was in turn clamped at Q to the centre of a horizontal bar V V. The bar V V consisted of a piece of thin, hollow, drawn brass tubing, of which the length was 30·70 centims. and the diameter 1·420 centim. This bar was graduated into millimetres and carried two suspenders, S, S, which were clamped to it at equal distances from the centre (fig. 3). The suspenders were each provided with an index such that their positions on the bar V V could be readily estimated to one-tenth of a millimetre. The mean diameter of the cylindrical portion, S K, of each suspender was 0·3366 centim., and the length of this portion 8·50 centims. To the ends, K, of the suspenders could be screwed (fig. 3) hollow cylinders of stiff paper or metal, or spheres of wood; when the former were employed the suspenders were provided with disks, *m*, *m*, of the same diameter as the cylinders, and about 2 millims. in thickness. Two brass caps, D, D (fig. 4), provided with screws about 8 centims. in length and 2 millims. in diameter, fit one into each end of the hollow bar V V, and can be easily removed from or placed in the latter.

To begin with, two cylinders or two spheres were screwed on to the ends of the suspenders (in the former case right up to the disks *m*, *m*), and the logarithmic decrement and the time of vibration determined from a very large number of vibrations. The cylinders or spheres were now unscrewed, and, the brass caps, D, D, having been temporarily removed for the purpose, two brass cylinders, *h*, *h* (fig. 4), each of the same mass as either of the vertical cylinders or spheres which had just been removed, were, by means of companion-screws, cut along their axes, adjusted on to the screws attached to the caps D, D, and at such a distance from the latter as preliminary experiments had proved would give nearly the same vibration-period, when the caps should be replaced in the bar V V, as had existed before the vertical

* 'Camb. Phil. Soc. Trans.,' vol. 9, No. X. (1850).

cylinders or spheres had been removed. The caps D, D, were now replaced in V V, and the logarithmic decrement, together with the time of vibration, was once more carefully determined. Observations such as these, when certain corrections presently to be mentioned had been applied, enabled one to calculate the effect of the resistance of the air on the vibrating vertical cylinders or spheres as far as the diminution of the amplitude of vibration was concerned.

The bar V V with its appendages was protected by a wooden box B of sufficient size to permit of vibrations, which, as regards the resistance of the air, were practically as free as in the open.* This box was provided with a window, E E, and two side-doors, lined with caoutchouc so as to fit air-tight; these side-doors were kept shut, except when it was necessary to make fresh adjustments. The torsional vibrations of the wire were observed by means of the usual mirror-and-scale arrangement, which is sufficiently shown in fig. 1, where M is the light mirror reflecting an illuminated circle of light crossed by a vertical, fine, dark line on to a scale bent into an arc of a circle of 1 metre radius, and placed at a distance of 1 metre from the mirror.

My three years' experience of the internal friction of metals had taught me that this last is by no means constant unless the greatest care be taken to prevent slight fluctuations of temperature. The above-mentioned fact seems to have escaped the notice of MAXWELL and MEYER, probably on account of the internal friction of the metal having a considerably less damping effect than the resistance of the air in their experiments. With me, however, especially in some cases, changes in the internal friction of the metal would have rendered it very difficult, nay, impossible, to attain the accuracy which I aimed at, and I deemed it advisable to protect the wire still further, as follows:—The top of the air-chamber W was well covered with baize, and surrounding W, and concentric with it, was a larger air-chamber X, made of tinned iron. This air-chamber was $11\frac{1}{2}$ inches in inner diameter, 15 inches in outer diameter, and 46 inches in height; the two concentric chambers of which it was composed enclosed between them a space 2 inches thick, stuffed with sawdust, whilst on the top of the chamber was placed a double cover A, also packed with sawdust. Passing through the outer air-chamber X, and through the walls of W, were two metal tubes in which were placed two thermometers T_1, T_2, with their bulbs near the wire; these thermometers were made by indiarubber tubing to slide air-tight in the metal tubes. A section of the two chambers X and W passing through one of the thermometers is shown in plan in fig. 2. The whole of this part of the apparatus rested on a stout wooden table, in which was pierced an aperture of a size just sufficient to allow the zinc tube Z, soldered to the air-chamber W, to pass through it and into the box beneath. A third thermometer T_3 served to give the temperature of the air in the

* In fig. 1 the cylinders appear to be closer to the sides of the box than they were in reality; the bar V V faced the window, but, for the sake of showing the arrangement of the cylinders better, it has been drawn facing the adjacent side of the box. The centres of the cylinders were at least six inches from the sides of the box.

box B, whilst the mean of the readings of T_1 and T_2 was used for the temperature of the wire. The thermometer T_3 was divided to one-tenth of a degree Centigrade, and had been tested at Kew; whilst the thermometers T_1 and T_2, which were graduated in degrees Centigrade, had been carefully compared by myself, degree by degree, with T_3.

The barometric pressure was registered by means of a delicate aneroid barometer, reading to $\frac{1}{100}$th of an inch, which has been in my possession for 15 years; this instrument I had recently compared with a standard mercury barometer.*

Before commencing the actual experiments on the viscosity of the air, it was found advisable to subject the wire to a preliminary training, in order not merely to diminish the internal friction of it, but also to make this last as constant as possible. In the first place, the wire was well annealed; this had the effect of reducing the internal friction of the hard-drawn metal to less than one-half of its previous amount.† In the next place, a load, equal to that of the cylinders or spheres to be used, having been suspended to VV, the wire was alternately heated to 100° C. and cooled again, this process being repeated for about a week, on each day of the week, until there was no further alteration of the internal friction of the wire when cool. This treatment still further reduced very considerably the damping of the vibrations due to the wire. The manner in which the heating was effected will be shown in a future paper, in which also will be recorded the results of experiments on the temporary effect of change of temperature on the torsional elasticity and internal friction of the metals used. When the wire had undergone this preliminary treatment, and all the arrangements were complete, the bar VV, with its appendant cylinders or spheres, as the case might be, was started by small impulses imparted by a worsted thread, until the arc of vibration, as reckoned from rest to rest, had reached about 400 divisions of the scale (about 10°, since 41·227 divisions represented 1°). After the arc of vibration from rest to rest had subsided to about 200 scale-divisions, the vibrator was again started, and this process was repeated until something like a thousand oscillations had been executed.‡ Finally the vibrator was re-started for the actual observations, through an arc of about 200 scale-divisions, and when about 50 oscillations had been executed after this last starting the readings were begun. Suppose that a_1, b_1; a_2, b_2; a_3, b_3; a_4, b_4; a_5, b_5, and a_6 are eleven consecutive

* In spite of the long period which has elapsed since this instrument was first made for me by the late Mr. BECKER, of ELLIOTT Bros., the spring still shows a slight amount of permanent yielding, which during the last two years has altered the reading by ·015 inch.

† Either silver, platinum, or copper wires, well annealed, may be used with advantage. I should not recommend unannealed piano-steel wire as used by MAXWELL; the last metal possesses, it is true, great elasticity, but the internal friction of silver, platinum, or copper can, by annealing, be made considerably less than that of the unannealed steel.

‡ The object of this treatment was to reduce the internal friction to its permanent condition, since long rest, or sometimes even a comparatively short rest, always raised sub-permanently the internal friction.

readings,* the ten corresponding arcs from rest to rest will be a_1+b_1, b_1+a_2, a_2+b_2, b_2+a_3, a_3+b_3, b_3+a_4, a_4+b_4, b_4+a_5, a_5+b_5, b_5+a_6. The means of a_1+b_1, b_5+a_6; b_1+a_2, a_5+b_5; a_2+b_2, b_4+a_5; b_2+a_3, a_4+b_4, and of a_3+b_3, b_3+a_4 were written down, and if these agreed well with each other, which was almost invariably the case, the logarithmic decrement of the mean of the five means was taken. Now, say that n single vibrations have taken place between the end of this and the beginning of the next set of consecutive readings, the difference between the logarithms of the first and second total means will, when divided by $n+10$, give the mean logarithmic decrement for a single vibration. The logarithmic decrement was found to be constant in each experiment within the limit of probable error; the deviations from uniformity were sometimes in one direction and sometimes in the opposite, and there was no evidence of any law of increase or diminution of the logarithmic decrement as the amplitudes decrease. In the intervals between one set and another of the readings, taken in the manner mentioned above, other readings were taken for the purpose of determining the vibration-period; the time of transit of the light across the centre of the scale, first in one direction and then in the opposite, was recorded for ten successive passages by means of a good watch provided with a seconds-hand, a similar series being recorded after every 200 vibrations. These last observations enabled the period of vibration to be determined with such exactness that we may completely disregard any error arising from want of precision in this respect. From time to time, at regular intervals, the readings of all three thermometers and of the aneroid barometer were taken, so that the mean pressure of the atmosphere, the temperature of the wire, and the temperature of the air in the box B could be calculated with the necessary accuracy. The greatest care was taken that the cylinders or spheres suspended from the horizontal bar VV should hang vertically; also that there should be no appreciable pendulous motion of the wire; if such motion existed it was checked by the hand before any of the readings were taken. Very great care was also taken in determining the moments of inertia of the vibrator in the various experiments, these being each obtained by several different methods,† which gave very concordant results. I shall have occasion in a future memoir to dwell on the various sources of error to which determinations of moments of inertia are liable; so it will suffice, perhaps, here to mention that this part of the work alone occupied my entire attention for nearly two weeks. The following five experiments, or rather sets of experiments, were made :—

Experiment I.

The wire was of well-annealed copper, 97 centims. in length and 0·06272 centim. in diameter. Two cylinders, each having a mass of 70·19 grammes, were used. These

* This number was always taken.

† The moments of inertia could be calculated with sufficient accuracy from the dimensions and mass of the vibrating system; they were, however, determined also indirectly by the two methods employed by MAXWELL.

cylinders were made of paper wrapped round a metal core a sufficient number of times to secure the requisite stiffness; the different layers of paper were pasted together, and when the whole was dry the metal core was withdrawn; the outside of each of the cylinders was also coated with French polish to prevent the absorption of moisture. The mean diameter of each of the cylinders was measured by calipers reading to $\frac{1}{1000}$th of an inch, and estimated to $\frac{1}{10000}$th of an inch. In obtaining the value of the mean diameter of each cylinder, twenty measurements were made, ten at equal intervals along the whole length, and ten at the same intervals, but in a direction at right angles to the first. The measurements showed a very fair uniformity of diameter throughout the whole length, the mean being 1·0079 inches for one cylinder and 1·0108 inches for the other. In the calculations subsequently made it was assumed that the diameter of each cylinder was the mean of the two last given, *i.e.*, was 1·0093 inches or 2·5636 centims. The lengths of the two cylinders were also very nearly the same, being 60·90 centims. and 60·85 centims. respectively; accordingly each cylinder was assumed to have a length of 60·875 centims. The ends of the cylinders consisted of wooden disks, into the centre of which was let a small brass disk provided with a screw, which was a companion to the screws at the ends of the suspenders S, K, so that the cylinders could be screwed right up to the disks M, M (fig. 3). The object of having the disks M, M, was to eliminate the effect of the friction of the air about the ends of the cylinders,* for Professor STOKES's mathematical investigations only apply strictly to cylinders of infinite length.

After the preliminary precautions previously mentioned had been taken the logarithmic decrement was determined from a great number of vibrations with the cylinders on; the cylinders were then each turned round their axes through a right angle, for the purpose of eliminating any error which might otherwise arise from the section of the cylinder being slightly elliptical instead of circular, and the logarithmic decrement was once more found. The cylinders were now unscrewed from the suspenders, and, the brass caps having been for the purpose removed from the hollow bar VV, the two brass cylinders *h*, *h*, were adjusted in the manner before mentioned, so that the vibration-period might remain very nearly unaltered; the caps were then replaced. All the adjustments alluded to above were performed very carefully so as to avoid jarring the wire, for if this precaution be not taken the internal friction will be temporarily increased, and will not come back to its previous value until the wire has been vibrated for a considerable time. A period of more than an hour was now allowed to elapse, the wire during this time being kept more or less in a state of vibration, but not through a greater arc than that represented by 400 scale-divisions from rest to rest, when the logarithmic decrement was again determined.

* It would have been well to have had these disks much thicker. As it is, the disks would only imperfectly serve the purpose for which they were intended; the effect about the ends of the cylinders was, however, completely eliminated in Experiment IV. It would appear, moreover, from the results that with the long cylinders here used the effect mentioned above is neglectable.

These processes were repeated during some eight or nine hours of each day through a period of three days, with the following *mean** results :—

PAPER Cylinders on.

Temperature of the air in degrees Centigrade.	Temperature of the wire.	Barometric height in inches.	Period of a single vibration in seconds.	Logarithmic decrement for one vibration.
12·02	12·43	29·872	6·8373	·0036476†
PAPER Cylinders off.				
12·25	12·31	29·817	6·8202	·0009103

The moment of inertia of the whole vibrator when the paper cylinders were on was 33773 in centimetre-gramme units.

Mathematical Formulæ necessary for the Investigation.

Before it can be shown how the results given above were made use of in finding the coefficient of viscosity of air, it will be necessary to point out how the requisite mathematical formulæ can be obtained. I will first take the case of a cylinder vibrating horizontally under the influence of the torsional elasticity of a wire attached to its centre and hanging vertically.

Conceive the cylinder divided into elementary slices by planes perpendicular to its axis. Let r be the distance of any slice from the middle point, θ the angle between the actual and the mean positions of the axis, $d\text{F}$ that part of the resistance experienced by the slice which varies as the first power of the velocity. Then, calculating the resistance as if the element belonged to an infinite cylinder moving with the same linear velocity, we have by Art. 31 of Prof. STOKES's paper—

$$d\text{F} = \frac{k'\text{M}'\pi}{\tau}\frac{d\xi}{dt},$$

where M′ is the mass of fluid displaced by the slice, $\frac{d\xi}{dt} = r\frac{d\theta}{dt}$, $\tau =$ the vibration-period, and k' is a constant, provided the vibration-period, the diameter of the cylinder, and the nature of the fluid remain unchanged.

* I have not thought it necessary to give here more than the mean values, as in a portion of a paper on the internal friction of metals, which I hope shortly to be able to offer to the Royal Society, I have entered fully into the details of experiments very similar to these.

† Mean of eight trials, each of 200 vibrations, the numbers varying from ·0036300 to ·0036969.

Let G be the moment of the resistance, l the whole length of the cylinder, a the radius of the cylinder, and ρ the density of the fluid; then

$$M' = \pi\rho a^2 dr,$$

and

$$G = \frac{\pi^2 k' \rho a^2 l^3}{12\tau}\frac{d\theta}{dt};$$

whence

$$\log_e \text{dec.} = \frac{\pi^2 k' \rho a^2 l^3}{24I},$$

I being the moment of inertia of the whole vibrator; thus

$$\log_{10} \text{dec.} = \frac{\pi^2 k' \rho a^2 l^3}{24I} \log_{10} e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When we have a pair of cylinders of equal mass and dimensions suspended vertically from points equally distant from the axis of the wire, we can easily prove in a manner similar to the above that the logarithmic decrement due to the resistance of the air on the cylinders is expressed by the formula

$$\log_{10} \text{dec.} = \frac{\pi^2 \rho \beta^2 l d^2 k'}{16I} \log_{10} e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)^*$$

If the logarithmic decrement be known, we can determine from (2) k', and hence, by interpolation, from the table given on p. 46 of Prof. STOKES's paper, $\mathfrak{m}$, this last being connected with μ, the coefficient of viscosity, by the formula—

$$\mathfrak{m} = \frac{\beta}{4}\sqrt{\frac{\pi\rho}{\tau\mu}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since β, τ, and ρ are known, we can from (3) find μ.

In the case of two spheres of equal mass and dimensions there is no difficulty in obtaining the following formulæ from the data on p. 32 of Prof. STOKES's paper:—

$$\log_{10} \text{dec.} = \frac{\pi k' M' d^2}{4(I + 2kM')} \log_{10} e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where I is the moment of inertia of the whole vibrator, M′ the mass of fluid displaced by each sphere, and k and k' are connected with μ by the equations—

* In this equation and in equation (4) the effect of the rotation of the cylinders about the axes is neglected. For the necessary correction see the end of the paper.

$$k = \tfrac{1}{2} + \frac{9}{4a}\sqrt{\frac{2\mu\tau}{\pi\rho}}, \quad \ldots \quad (5)$$

$$k' = \frac{9}{4a}\sqrt{\frac{2\mu\tau}{\pi\rho}}\left\{1 + \frac{1}{a}\sqrt{\frac{2\mu\tau}{\pi\rho}}\right\}, \quad \ldots \quad (6)$$

in which a is the radius of each sphere.

Application of the Mathematical Formulæ to the Results of Experiment I.

It will be seen that the logarithmic decrement with the paper cylinders on is ·0036476, whilst with the paper cylinders off it is ·0009103; therefore the logarithmic decrement due to the resistance of the air on the cylinders only is approximately ·0027373. I write 'approximately' because there are certain corrections to be applied which I will now proceed to describe. In the first place, the vibration-period, when the paper cylinders were on, though nearly the same as when the cylinders were off, was not quite the same. I therefore determined approximately the value of μ, without making this or the other small corrections to be mentioned presently, and used this value to obtain approximately the logarithmic decrement which would be due to the resistance of the air on the cylindrical bar VV and the cylindrical portions S, K, of the suspenders. The logarithmic decrement due to the resistance of the air on the other portions of the suspenders and on the disks m, m, was obtained by making independent observations, in which the bar was vibrated first with the suspenders on the bar, and then with the suspenders off, but with cylinders of equal mass placed inside the hollow bar VV, so that the time of vibration should remain unaltered.

Suppose that λ represents the logarithmic decrement due to the resistance of the air on the bar and the suspenders, and that t_1, t_2, are the vibration-periods with and without the paper cylinders respectively, then, with a sufficient degree of approximation, provided t_1 does not differ much from t_2, we have the amount to be added to the uncorrected logarithmic decrement equal to

$$\lambda\left(1 - \frac{t_2^{\frac{3}{2}}}{t_1^{\frac{3}{2}}}\right).$$

Again, the temperature of the air and the pressure of the atmosphere were not quite the same with and without the paper cylinders. It can, however, be shown that for the small differences of temperature and pressure which we have here the logarithmic decrement will be independent of the temperature* and vary directly

* The logarithmic decrement will not be independent of the temperature unless μ varies as the absolute temperature. If we adopt the results of recent experiments, the logarithmic decrement should approximately vary as $\sqrt{\frac{364+t}{273+t}}$, where t is the temperature in degrees Centigrade. The correction which this would entail I have neglected, as being inappreciable in these experiments.

as the square root of the pressure; the amount to be added to the uncorrected logarithmic decrement, owing to the above causes, will therefore be

$$\lambda\left(1-\sqrt{\frac{p_1}{p_2}}\right),$$

where p_1 and p_2 are the pressures with and without the paper cylinders respectively.

Further, when the cylinders were screwed on to the suspenders, about 4 mms. of the latter entered the former, so that the observed logarithmic decrement was less than it should be by an amount which would be nearly equal to the logarithmic decrement due to the resistance of the air on two vertical cylinders 4 mms. in length and 0·3366 cm. in diameter; this could be calculated to within a sufficient degree of approximation by using the approximate value of μ. The amount in this particular case was ·0000037.

Lastly, the temperature of the wire was not the same with and without the paper cylinders, but, as the effect of change of temperature had been determined previously, this difference could be allowed for.

No correction is required for any variation in the internal friction of the wire itself, arising from difference in the vibration-periods with and without the paper cylinders; for I had previously satisfied myself that the diminution of amplitude resulting from internal friction is nearly independent of the time of vibration.

Accordingly we have the following amounts to be added to the uncorrected logarithmic decrement:—

	Correction.
For difference of time of vibration with and without paper cylinders	+·0000008
For difference of pressure of air	—·0000005
For difference of temperature of the wire	—·0000002
For portions of suspenders which enter the cylinders	+·0000037
Total	+·0000038
Corrected logarithmic decrement	·0027411

In calculating ρ, the density of the air, I have assumed that the latter is half saturated with moisture, and that the mass of a cubic centimetre of dry air at 0° C., and under a pressure of 29·9217 inches of mercury, is ·0012930 gramme; thus, in the present instance,

$$\rho=\frac{29{\cdot}872-\frac{3}{8}\times{\cdot}206}{29{\cdot}9217}\times\frac{273}{273+12{\cdot}02}\times{\cdot}001293={\cdot}0012334.$$

The distance from each other of the axes of the two paper cylinders was 20·80 centims., and this distance was maintained in all the experiments which follow, except the last, where it was 20·78 centims.

From these and the previous data we can, by means of equation (2), get

$$k' = 1{\cdot}6122\,;$$

and hence, by interpolation, we can obtain from the table on page 46 of Professor STOKES's paper

$$\mathfrak{m} = 1{\cdot}1327.$$

Again, substituting this value of $\mathfrak{m}$ in equation (3), we obtain as the value of μ in C.G.S. units, at the temperature of 12°·02 C,

$$\cdot 00018294.$$

Experiment II.

Two hollow cylinders, made of drawn brass tubing, and closed at both ends, were used instead of the paper cylinders. As measured by a gauge reading to $\frac{1}{100}$th of a millimetre, the mean diameter of one cylinder was 0·96446 centim., and of the other 0·96279 centim. These values were obtained by gauging each cylinder in ten different places, equidistant from each other, and in the calculations each cylinder was assumed to have a mean diameter of 0·96363 centim. The length of one cylinder was 60·92 centims., and of the other 60·85 centims., whilst the mean of these numbers, *i.e.*, 60·885 centims., was assumed to be the length of each cylinder. The mass of each cylinder was 91·900 grammes, and when the cylinders were on the bar V V the moment of inertia of the whole vibrator, in centimetre-gramme units, was 36702. The value of the vibration-period was 7·0590 seconds. The temperature of the air was 14°·63 C., and the barometric height 29·707 inches. The uncorrected logarithmic decrement due to the resistance of the air on the cylinders was ·0012338, and the corrected logarithmic decrement was ·0012546. From these data was deduced a value of μ, at the temperature of 14°·63 C., of

$$\cdot 00017718.$$

Experiment III.

Everything else was arranged in the same manner as in Experiment I., but, instead of the annealed copper wire, an annealed silver wire, 97 centims. in length and 0·100863 centim. in diameter, was used. The paper cylinders employed in Experiment I. were used here, and when these cylinders were on the vibration-period was 3·0198 seconds. The temperature of the air was 11°·69 C., and the barometric height 30·207 inches. The uncorrected logarithmic decrement due to the resistance of the air against the cylinders was ·0016871, and the corrected logarithmic decrement ·0016905. The value of μ at the temperature 11°·69 C. was calculated to be

$$\cdot 00018143.$$

Experiment IV.

Acting on the advice of Professor STOKES, I modified Experiment III. as follows:—The logarithmic decrement was determined with the paper cylinders already used, and also with another pair of the same diameter, and made in the same manner, but having a length of 7·700 centims., the vibration-period being by the usual device maintained very nearly the same in both cases. The difference between the two logarithmic decrements, ·0024564 and ·0009933, will therefore equal the logarithmic decrement due to the resistance of the air on cylinders having each a length of (60·875–7·700) centims., *i.e.*, 53·175 centims. When the longer paper cylinders were on the bar the vibration-period was 2·9994 seconds. The temperature of the air and the barometric height were 10°·64 C. and 30·057 inches respectively. The uncorrected logarithmic decrement was ·0014631, and the corrected logarithmic decrement ·0014638. The value of μ at the temperature of 10°·64 C., deduced from the above data, was

·00017955.

Experiment V.

The previous experiments had given such closely according values of μ that, though my investigations on the internal friction of metals only required that the formulæ for cylinders should give consistent results, I felt that it would be of interest to ascertain whether the use of spheres would be attended with the same satisfactory agreement. The main difficulty to be encountered with spheres is that the mass of a properly constructed spherical shell makes it rather unsuitable for experiments on the viscosity of gases. After thinking over various plans of obtaining hollow spherical shells of sufficiently accurate make, and not feeling satisfied that I should be able to get, without much difficulty, what I wanted, I decided on using solid spheres made of fairly light wood. These spheres were specially turned for me, with instructions to make each as exactly as possible $2\frac{1}{2}$ inches in diameter. The turner executed his commission very fairly, for, on gauging each sphere at ten different places with calipers reading to $\frac{1}{1000}$th of an inch, I found that none of the readings differed from the mean by so much as ·3 per cent., and that the mean diameters of the two spheres were 2·5103 inches and 2·5007 inches respectively. In the calculations each sphere was reckoned as having a diameter of 2·5055 inches or 6·364 centims. The masses of the two spheres were not quite so equal as I could have wished, the apparent mass of one in air being 64·823 grammes, and of the other 63·761 grammes. No appreciable error will, however, be introduced by considering the apparent mass of each in air to be 64·292 grammes. The correction for the mass of air displaced by each sphere amounted to 0·168 gramme, so that in the calculations the mass of each sphere was taken as 64·460 grammes.

The spheres were attached to the suspenders S, K, in the same manner as the cylinders, but the disks were now dispensed with. The moment of inertia of the whole vibrator when the spheres were on was 30,927 in centimetre-gramme units, the vibration-period was 2·8791 seconds, and the temperature of the air and the barometric pressure were 9°·97 C. and 29·607 inches respectively. The uncorrected logarithmic decrement due to the friction of the air on the spheres was ·0003462, and the corrected logarithmic decrement was ·0003483.

In deducing the value of μ from the above data by the aid of equations (4), (5), and (6), I assumed, in finding $2k\mathrm{M}'$, a value for μ equal to the mean of that got from the other experiments; this step is admissible, because $2k\mathrm{M}'$ is very small compared with I.* Having determined the value of k' by means of equation (4), I substituted it in equation (6), and thus obtained a quadratic equation for finding μ. The quadratic may, however, be converted into a simple equation by making use of the same value of μ as above in calculating the term $\frac{1}{a}\sqrt{\frac{2\mu\tau}{\pi\rho}}$, which was thus found to be 0·16085. The last number is not small compared with unity, and, had the final result proved to be as much as 10 per cent. greater or less than the mean of those got from the other experiments, the above conversion of the quadratic into the simple equation would not have been admissible. It will be seen eventually, however, that the conversion is legitimate, and the value of μ at a temperature of 9°·97 C. as determined from the simple equation is

·00019334.

Mathematical Formulæ required for the Effect of the Rotation of the Spheres or Cylinders about their own Axes.†

Professor G. G. STOKES has been good enough to furnish me with the following formulæ for the corrections not yet made for the effect of the rotation of the spheres or cylinders about their own axes:—

Let λ_a be the logarithmic decrement due to the rotation, then for the spheres

$$\lambda_a = \frac{2\mu\mathrm{M}'\tau}{\mathrm{I}\rho}\,\frac{\nu a + 3 + \frac{3}{\nu a} + \frac{3}{2(\nu a)^2}}{1 + \frac{1}{\nu a} + \frac{1}{2(\nu a)^2}}\log_{10} e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where I is the moment of inertia of the whole system, τ is the time of a vibration

* In fact, is quite neglectable in the case before us.

† What follows was added Sept. 16, 1886.

from rest to rest, M′ is the mass of fluid displaced by each sphere, a is the radius of the sphere, and

$$\nu = \sqrt{\frac{\pi\rho}{2\mu\tau}}.$$

In the case of the cylinders, which were hollow, we have to take into account the effect of the air both inside and outside. For the air outside we may take

$$\lambda_a = \frac{2M'\mu\tau}{I\rho} \log_{10} e P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where P is the real part of the imaginary expression

$$ma \frac{1+\frac{3.5}{1(8ma)}+\frac{1.3.5.7}{1.2(8ma)^2}-\frac{1^2.3.5.7.9}{1.2.3(8ma)^3}+\frac{1^2.3^2.5.7.9.11}{1.2.3.4(8ma)^4}-\ldots}{1+\frac{1.3}{1(8ma)}-\frac{1^2.3.5}{1.2(8ma)^2}+\frac{1^2.3^2.5.7}{1.2.3(8ma)^3}-\ldots},$$

where

$$m = \sqrt{\frac{\pi\rho}{\mu\tau}} (\cos 45^\circ + \sqrt{-1} \sin 45^\circ).$$

On expanding P in descending powers of $\sqrt{\frac{\pi\rho}{\mu\tau}}\,a$, we get

$$P = \frac{f}{\sqrt{2}} + 1.5 + \frac{0{\cdot}375}{\sqrt{2}f} - \frac{0{\cdot}4922}{\sqrt{2}f^3} - \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$f = \sqrt{\frac{\pi\rho}{\mu\tau}}.a.$$

This series may be used with advantage in all the experiments relating to the cylinders to estimate approximately the effect of the air outside, but, unless the value of f is decidedly larger, the value of λ_a is best found from the formula

$$\lambda_a = \frac{4M'\mu\tau}{I\rho} \log_{10} e.\frac{k^2+k'^2-1}{(k-1)^2+k'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where k, k', are the quantities tabulated at p. 46 of Professor STOKES's paper.

The corrections, as calculated from both formulæ, were found to agree satisfactorily.

For the air inside we may use, for such values of $\sqrt{\frac{\pi\rho}{\mu\tau}}\,a$ as we have here, the formula

$$\lambda_a = \frac{2M'\mu}{I\rho} (-Q) \log_{10} e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where M′ is the mass of air inside, and Q is the real part of the imaginary expression

$$-\frac{\frac{2}{2.4}(ma)^2+\frac{4}{2.4^2.6}(ma)^4+\frac{6}{2.4^2.6^2.8}+\ldots}{1+\frac{1}{2.4}(ma)^2+\frac{1}{2.4^2.6}(ma)^4+\frac{1}{2.4^2.6^2.8}(ma)^6+\ldots},$$

which is of the form

$$\frac{A+\sqrt{-1}B}{C+\sqrt{-1}D}$$

and of which the real part is

$$\frac{AC+BD}{C^2+D^2}.$$

In the following table will be found the corrections necessary to be made for the rotations of the spheres and cylinders about their axes :—

CYLINDERS.

Number of experiment.	Logarithmic decrement uncorrected for rotation about axes.	Effect of air outside. *	Effect of air inside. *	Corrected logarithmic decrement.
I.	·0027411	·0000313	·0000011	·0027087
II.	·0012546	·0000030	·0000000	·0012516
III.	·0016905	·0000173	·0000019	·0016713
IV.	·0014638	·0000150	·0000017	·0014471
		Spheres.		
V.	·0003483	·0000159	..	·0003324

There is still a further slight correction to make, inasmuch as the mercury of the barometer was not at 0° C. when the aneroid was compared with the mercury barometer, whereas the density of the air was calculated on the assumption of the mercury being at 0° C. The correction is very slight, but the closeness of agreement of the different experiments justifies us in making it. It will be sufficient for this purpose to multiply each value of μ as determined from the above table by $(1+\cdot00018t)$, where t is the temperature at which the aneroid was compared with the mercury barometer. Applying all the corrections, the final results are as follows :—

* In making these corrections, an approximate value of μ was used.

TABLE II.—Cylinders.

Number of experiment.	Length in centims.	Diameter in centims.	Distance between the centres in centims.	Vibration-period in seconds.	Temperature in degrees Centigrade.	Coefficient of viscosity of air in C.G.S. units.
I.	60·875	2·5636	20·80	6·8373	12·02	·00017900
II.	60·885	0·9636	20·80	7·0590	14·63	·00017680
III.	60·875	2·5636	20·80	3·0198	11·69	·00017767
IV.	53·175	2·5636	20·80	2·9994	10·64	·00017581
Spheres.						
V.	..	6·364	20·78	2·8811	9·97	·00017626

Taking the means of the numbers in the sixth and seventh columns, we find that the value of μ at a temperature of 11°·79 C. is

$$·00017711.$$

The Effect of the Presence of Aqueous Vapour on the Viscosity of Air.

The above experiments extended over a period of some months, during which the air was in various conditions with respect to being saturated with aqueous vapour, so that for a rough approximation we may assume that the mean value for μ just given will apply to air half saturated with vapour at a temperature of 12° C., and it would appear that the presence of the small quantity of aqueous vapour which this implies would not affect the value of μ to an extent equal to that of the probable error in experimenting. From the careful investigations of Mr. CROOKES* we learn that at a temperature of 15° C., and under pressures of from 760 to 350 millims., the presence of aqueous vapour has little or no influence on the logarithmic decrement. By the aid of Professor STOKES's note,† I have estimated that at 15° C., and under a pressure of 760 millims., the air when *saturated* with aqueous vapour would be *more* viscous than perfectly dry air ‡ to the extent of only ·2 per cent. It is not until the air is under a less pressure than 350 millims. that the aqueous vapour begins to show appreciable effect, but when the rarefaction is great the moist air becomes considerably *less* viscous than dry air.

According to MAXWELL § damp air over water at a temperature of 21°·11 C., and under a pressure of 101 millims., is *less* viscous than dry air by about $\frac{1}{60}$th part.

* 'Phil. Trans.,' Part II., 1881, p. 427.

† See p. 440 of the above paper.

‡ Mr. CROOKES adopted great precautions to render the air dry.

§ 'Phil. Trans.,' vol. 156, 1866.

On the whole it would seem that the aqueous vapour in the air used in my experiments would hardly influence the value of μ to the extent of ·1 per cent.

The presence of carbon dioxide in the air would still less affect the result, as not only is the viscosity of carbon dioxide not very remote from that of air, but the amount of the gas present is also very minute.

Comparison of the Results of Recent Investigations of the Coefficient of Viscosity of Air.

In the beginning of this memoir I pointed out the very large discrepancies which existed between the results of different experimenters, but, since I entered on my task, not only have I acquired fresh information respecting what had already been done, but also quite recently fresh investigations have been made. Table III. contains the required information.

TABLE III.

Authority.	Coefficient of viscosity of air at 0° C.	Method.
O. E. MEYER*	·0001875	Oscillating plates.
"	·0001727	Transpiration.
PULUJ*	·0001798	"
SCHNEEBELI†	·0001707	"
OBERMAYER†	·0001705	"

In order to reduce my own observations to 0° C., I have made use of the investigations of Professor SILAS W. HOLMAN on the effect of temperature on the viscosity of air.‡ According to the exceedingly careful and elaborate observations of this experimenter, the coefficient of viscosity of dry air is not proportional to the absolute temperature, but

$$\mu_t = \mu_0(1+0·002751t - 0·00000034t^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where t is the temperature in degrees Centigrade, and μ_t, μ_0, are the coefficients of viscosity at t° C. and 0° C. respectively.

My own observations were made with too small ranges of temperature to show the relation between the value of μ and the temperature, but the above formula expresses more nearly this relation as deduced from my experiments than the formula

$$\mu_t = \mu_0(1+0·00366t).$$

* 'Phil. Mag.,' vol. 21, 1886, p. 220.

† 'Archives Sci. Phys. Nat.,' vol. 14, 1885.

‡ 'Phil. Mag.,' vol. 21, 1886.

Adopting, therefore, formula (12), we have the following equation for determining the value of μ at any temperature :—

$$\mu_t = \cdot 00017155(1 + \cdot 002751t - \cdot 00000034t^2). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The differences between the observed and calculated values of μ_t for the five different sets of experiments are given below :—

Experiment.	Observed value of μ_t.	Calculated value of μ_t.	Difference.
I.	·00017900	·00017760	+·00000140
II.	·00017680	·00017850	−·00000170
III.	·00017767	·00017704	+·00000063
IV.	·00017581	·00017653	−·00000072
V.	·00017626	·00017622	+·00000004

The probable error is about ·2 per cent., and, considering the manner in which the five sets of experiments varied as regards their conditions, it would seem that, even when all allowance has been made for aqueous vapour, &c., the number ·00017155 must represent the value of μ_0 for *dry* air within at least $\frac{1}{2}$ per cent. Now, this number agrees fairly with the values of μ_0 obtained by other observers with the transpiration method; it is, however, more than 9 per cent. less than that obtained by MEYER with oscillating plates, and by MAXWELL. The mathematical difficulties attending Professor MEYER'S method of oscillating plates have been already mentioned, but the method of Professor MAXWELL does not seem open to these objections, and indeed appeared to me to be so good that I for some time attempted, though in vain, to account for the difference between MAXWELL'S result and my own. Professor G. G. STOKES has, however, kindly interested himself in the matter, and has shown in the accompanying note the possibility of MAXWELL'S result being too high. I may perhaps be allowed to add that, if we only take the first two of the five sets of MAXWELL'S experiments, in which two the distances of the fixed from the oscillating plates are so great as to render any error such as suggested by Professor STOKES very small, we obtain a value for the coefficient which is nearly identical with that obtained by myself.

ADDENDUM.

Note on the preceding Paper, by Professor G. G. STOKES, *P.R.S.*

(Received January 14, 1886.)

The consistency of Mr. TOMLINSON'S different determinations of the coefficient of viscosity of air, notwithstanding the great variation in the circumstances of the experiments, and the consistency with one another of the numbers got by a different process by MAXWELL, led me to endeavour to make out the real cause of the difference, and I think the main part, at any rate, of it can be explained by a very natural supposition.

The fact that Mr. TOMLINSON worked with air in its ordinary state, whereas MAXWELL'S air was dry, even if it tends in the right direction, would evidently not go nearly far enough. But it occurred to me that the effect of any error of level in the movable disks employed by MAXWELL must have been much greater than might at first sight appear. For suppose a very small error δ to exist, and suppose the fixed disks adjusted to be parallel to the movable ones in the position of equilibrium of the latter. Then the two systems must be, very nearly indeed, parallel throughout the motion, since the angle of oscillation of the movable disks to one side or other of the position of equilibrium is very small. If 2α be the whole amplitude, the greatest error of parallelism will be of the order $\delta\alpha$, and it would naturally appear at first sight that the effect of so small an error of parallelism must be insignificant for any such error of level as we can reasonably suppose to have existed. But a little consideration will show that this need not be the case when the distance between the fixed and movable disks is very small compared with the diameter of the latter. For suppose the disk to have been rotated through a small angle ρ round a vertical axis; the rotation ρ may be decomposed into a rotation $\rho \cos \delta$ round the axis of figure, and a rotation $\rho \sin \delta$ round a horizontal axis in the plane of the disk. As regards the former, the motion takes place as supposed in the investigation. But as regards the latter the disk oscillates about a horizontal axis in its own plane. Now, when the disks are very near one another this oscillation entails a squeezing thinner of the stratum of air opposite to one half of the disk, and a widening of the stratum opposite the other half, the two halves being alternately squeezed thinner and widened; and, since for such slow motions the air is practically incompressible, this transfer of air cannot be effected without a motion of the air along the surface of the disk far larger than what would be produced by an equal rotation about the axis of figure. Accordingly a very slight error of horizontality in the movable disk might produce a sensible error in the result, though an error of direction of similar amount in the orientation of the fixed disk would be quite insignificant in its influence on the final result.

This conclusion is fully borne out by the result of mathematical calculation founded on the equations of motion of a viscous uncompressed fluid. The calculation becomes very simple if we treat the distance between the disks as very small compared with the radius, neglect the special actions about the edge, and further neglect the inertia of the air, as we safely may, since it was small in MAXWELL'S experiments, especially those in which the disks were at a small distance apart, and therefore the influence of viscosity the greatest; or those again in which the air was rarefied.

Let the plane of a movable disk in its position of equilibrium be taken for the plane of x, y, the axis of figure for the axis of z, and the intersection of a horizontal plane with the plane of the disk for the axis of y; and let the opposed fixed plane be parallel to the plane of x, y, and at a distance h from it. Let a be the radius of the disk.

First, as regards motion round the axis of figure. Let ω be the angular velocity of the disk. Then, according to the simplifications adopted, the motion of the fluid will be a motion of simple shearing, such that the velocity at a point whose semi-polar coordinates are r, θ, z, will be $\omega r(h-z)/h$ in a direction perpendicular to the radius vector. It will suffice to write down the moment of the force which this calls into play, which is

$$\frac{\pi\mu a^4\omega}{2h}. \qquad \text{(A)}$$

Next, for motion round the axis of y. Let ω' be the angular velocity; u, v, w, the components of the velocity; U, V, the mean values of u, v, from 0 to h. Consider the prism of fluid standing on the base $dx\,dy$, and extending between the planes. As the volume of the prism is diminished at the base by $\omega' x\,dx\,dy\,dt$ in the time dt, the excess of the volume of the fluid which flows out across the face hdy, whose abscissa is $x+dx$, over that which flows in across the face hdy, whose abscissa is x, *plus* the similar difference for the pair of faces hdx, must equal $\omega' x\,dx\,dy\,dt$. This leads to the equation

$$h\frac{dU}{dx}+h\frac{dV}{dy}=\omega' x. \qquad (1)$$

But, for motion between two close parallel planes, the velocity parallel to the plane, and its components in two fixed directions in that plane, vary as $z(h-z)$, and therefore

$$u=\frac{6z(h-z)}{h^2}U, \qquad v=\frac{6z(h-z)}{h^2}V. \qquad (2)$$

The first equation of motion is

$$\frac{dp}{dx}=\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right). \qquad (3)$$

Now, on account of the smallness of h, the space-variations of the components u, v, of the velocity are much greater for z than for x or y. Hence in (3), and the corresponding equation for v, the first two terms in the right-hand members may be omitted, giving, by (2),

$$\frac{dp}{dx} = -\frac{12\mu}{h^2}\mathrm{U}, \qquad \frac{dp}{dy} = -\frac{12\mu}{h^2}\mathrm{V},$$

and then, from (1),

$$\frac{d^2p}{dx^2}+\frac{d^2p}{dy^2} = -\frac{12\mu\omega'}{h^3}x,$$

or, in polar coordinates,

$$\frac{d^2p}{dr^2}+\frac{1}{r}\frac{dp}{dr}+\frac{1}{r^2}\frac{d^2p}{d\theta^2} = -\frac{12\mu\omega'}{h^3}r\cos\theta\,; \quad \ldots \ldots \ldots \quad (4)$$

and if we take, as we may, p to mean the excess of pressure over the pressure in equilibrium, we have the conditions that p shall vanish when $r=a$, and that p shall not become infinite at the centre.

The equation (4) and the conditions at the mouth and centre may be satisfied by taking

$$p = f(r)\cos\theta,$$

which gives, from (4),

$$f''(r)+\frac{1}{r}f'(r)-\frac{1}{r^2}f(r) = -\frac{12\mu\omega'}{h^3}r.$$

The integral of this equation is

$$f(r) = -\frac{3\mu\omega'}{2h^3}r^3+\mathrm{A}r+\frac{\mathrm{B}}{r},$$

where A, B, are arbitrary constants. The conditions at the centre and mouth give

$$\mathrm{B} = 0, \qquad \mathrm{A} = \frac{3\mu\omega' a^2}{2h^3},$$

whence

$$p = \frac{3\mu\omega'}{2h^3}(a^2r-r^3)\cos\theta.$$

The moment of this pressure about the axis of y is $\iint p.r\cos\theta.rdrd\theta$, or

$$\frac{\pi\mu\omega' a^6}{8h^3}. \quad \ldots \ldots \ldots \quad (\mathrm{B})$$

The moments (B) and (A) are as $a^2\omega'$ to $4h^2\omega$, and the works of these moments in the time dt are as $a^2\omega'^2$ to $4h^2\omega^2$. If this ratio be denoted by n to e, and ω, ω', are the components of an angular velocity round an axis in the plane of xz, inclined at an angle δ to the axis of z,

$$\tan^2\delta = \frac{4h^2}{a^2}n.$$

In MAXWELL'S experiments a was 5·28 inches, and when the fixed and movable disks were closest h was 0·18475. If we suppose the whole loss of energy 8 per cent. greater than that due to rotation round the axis of figure, to which it was deemed to be due, we have $n=0·08$, giving $\delta=1° 8'$. Now, no special adjustment was made to secure the strict horizontality of the movable disks, or at least none is mentioned; the final adjustment is stated to have been that of the fixed disks, which were presumably adjusted to be parallel to the movable ones, and at the desired distance. Hence such small errors of level as that just mentioned may very well have occurred.

Second Note.—*On the Effect of the Rotations of the Cylinders or Spheres round their own Axes in increasing the Logarithmic Decrement of the Arc of Vibration.*—By the same.

(Received October 22, 1886.)

In Art. 9 of my paper on Pendulums I pointed out that in the case of a ball pendulum the resistance due to the rotation of the sphere round its axis need not be regarded, on account of the large ratio which the distance of the centre from the axis of suspension bears to the radius of the sphere. In Mr. TOMLINSON'S experiments the corresponding ratio is not near so great, and its squared reciprocal is not small enough to allow us to neglect the correction altogether, especially in the case of the spheres, the radius of which is much larger than that of the cylinders. In both cases the problem admits of solution.

In both cases the motion of the suspended body may be regarded as compounded of a motion of translation, in which the centre oscillates in an arc of a circle, and a motion of rotation about its axis of figure, supposed fixed; and, the motion being small, the effects of the two may be considered separately. It is the latter with which we have at present to deal. As regards the motion of translation, the spheres or cylinders were sufficiently far apart to allow us to regard each as out of the influence of the other, and accordingly as oscillating in an infinite mass of fluid; and this is still more nearly true as regards the motion of rotation. The problem, then, is reduced to this: a sphere or cylinder performs small oscillations of rotation about its axis of figure, which is vertical and regarded as fixed, in an infinite mass of viscous fluid; it is required to determine the motion, and thereby to find the effect of the fluid in damping the motion of the system of which the suspended body forms a part.

In the case of the sphere the problem of determining the motion of the fluid is identical with that solved by Professor VON HELMHOLTZ in a paper published in the 40th volume of the 'Sitzungsberichte' of the Vienna Academy, p. 607, and reprinted in the first volume of his collected works, p. 172, with the exception that the arbitrary constants which occur in the integral of the fundamental ordinary differential equation are differently determined, since the condition that the motion shall not become infinite at the centre is replaced by the condition that it shall not be infinite at an infinite distance.

In the present case the motion is necessarily symmetrical about the axis, so that it is alike all round any circle that has the axis for its axis; it is, moreover, tangential to the circle. Let the fluid be referred to polar coordinates r, θ, ϖ; r being the distance from the centre, θ the inclination of the radius vector to the axis, and ϖ as usual. Then, taking ρ, μ, to denote the density and coefficient of viscosity, and observing that $v=q\cos\varpi$, where q is the velocity, we easily get from the second equation of motion, by putting, as we may, $\varpi=0$ after differentiation,

$$\frac{d^2q}{dr^2}+\frac{2}{r}\frac{dq}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dq}{d\theta}\right)-\frac{q}{r^2\sin^2\theta}-\frac{\rho}{\mu}\frac{dq}{dt}=0\,; \quad . \quad . \quad . \quad (1)$$

and we have the condition at the surface—

$$q=\omega a\sin\theta \quad \text{when} \quad r=a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ω is the angular velocity of the sphere, and a its radius.

The motion with which we have to deal is periodic, subject to a secular diminution. The latter being actually very slow, it will suffice, in calculating the force of the air on the sphere, to take the motion as periodic, and expressed, so far as the time is concerned, by the sine or cosine of nt. It will be more convenient, however, to use the symbolical expression e^{int}, where $i=\sqrt{(-1)}$. The general equation (1) and the equation of condition (2) can both be satisfied by taking q to be expressed, so far as θ is concerned, by $\sin\theta$. Assuming, then—

$$q=e^{int}\sin\theta f(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and writing

$$\frac{i\rho n}{\mu}=\frac{i\pi\rho}{\mu\tau}=m^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have

$$f''(r)+\frac{2}{r}f'(r)-\frac{2}{r^2}f(r)-m^2f(r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Taking $+m$ for that root of the imaginary m^2 which has its real part positive, we have for the integral of (5), subject to the condition of not becoming infinite at an infinite distance—

$$f(r)=\mathrm{A}e^{-mr}\left(\frac{1}{r}+\frac{1}{mr^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Omitting the pressure in equilibrium, we shall have for the force of the fluid on an element of the sphere a tangential pressure (say T, referred to a unit of surface) acting perpendicularly to the plane passing through the axis and the element, the expression for which, reckoned positive when it acts in the direction of ϖ positive, is

$$\mathrm{T}=\mu\left(\frac{dq}{dr}-\frac{q}{r}\right)_{r=a}=\mu e^{int}\sin\theta\left(f'(a)-\frac{f(a)}{a}\right);$$

and the moment of the force taken all over the sphere is

$$\int_0^\pi \mathrm{T}\ 2\pi a^2\sin\theta . a\sin\theta\, d\theta=\tfrac{8}{3}\pi\mu a^3 e^{int}\left(f'(a)-\frac{f(a)}{a}\right)$$

$$=2\mathrm{M}'\mu' e^{int}\left(f'(a)-\frac{f(a)}{a}\right)$$

if $\mu'=\mu/\rho$, and M′ is the mass of the fluid displaced by the sphere.

Now we have, by (2), (3),

$$\omega a=e^{int}f(a),$$

whence the expression for the moment becomes

$$2\mathrm{M}'\mu'\left(\frac{af'(a)}{f(a)}-1\right)\omega.$$

To get the whole moment, the above must be doubled, as there are two spheres. If ϑ be the angular distance of the vibrating system from the position of equilibrium, we may write $d\vartheta/dt$ for ω; and if the mixed imaginary within parentheses, with sign changed, be denoted by $\mathrm{P}+i\mathrm{Q}$, the real part, P, will be that which affects the arc of vibration, the imaginary part falling upon the time, which we do not want. The Napierian logarithmic decrement in one vibration will be got by dividing half the real part of the expression for the moment of the forces by the moment of inertia, or, say, MK^2. It will therefore be $2\mathrm{M}'\mu'\mathrm{P}/\mathrm{MK}^2$.

Now we get, from (6),

$$1-\frac{af'(a)}{f(a)}=\frac{ma+3+\dfrac{3}{ma}}{1+\dfrac{1}{ma}};$$

and, taking the real part of this, we get finally, after reduction,

$$\text{Nap. log. dec.}=\frac{2\mu'\mathrm{M}'}{\mathrm{MK}^2}\,\frac{\nu a+3+\dfrac{3}{\nu a}+\dfrac{3}{2\nu^2a^2}}{1+\dfrac{1}{\nu a}+\dfrac{1}{2\nu^2a^2}}\tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

where $\nu=\sqrt{\left(\dfrac{\pi}{2\mu'\tau}\right)}$.

In the case of the cylinder the motion is in two dimensions, and is most conveniently referred to polar coordinates r, θ, the origin being in the axis. The radius of the cylinder will be denoted by a, the outer or inner radius according as we are dealing with the air outside or inside.

The mode of proceeding is precisely analogous to that in the case of the sphere, and, q being the whole velocity, we have

$$q = e^{int} f_1(r), \quad \ldots \quad (8)$$

where

$$f_1''(r) + \frac{1}{r} f_1'(r) - \frac{1}{r^2} f_1(r) - m^2 f_1(r) = 0; \quad \ldots \quad (9)$$

and the condition at the surface gives

$$e^{int} f_1(a) = \omega a. \quad \ldots \quad (10)$$

If T be the tangential pressure on the cylinder,

$$\mathrm{T} = \pm\mu\left(\frac{dq}{dr} - \frac{q}{r}\right)_{r=a}, \quad \ldots \quad (11)$$

the sign being + or − according as we are dealing with the air outside or inside. The moment of this pressure on a length, l, of the cylinder is

$$\pm 2\pi\mu a^2 l e^{int}\left(f_1'(a) - \frac{1}{a} f_1(a)\right) = \pm 2\mathrm{M}'\mu'\left(\frac{af_1'(a)}{f_1(a)} - 1\right)\omega. \quad \ldots \quad (12)$$

The equation (9) cannot be integrated in finite terms. Nevertheless, in the case of the air outside, the expression (12) for the moment may be obtained in a finite form in terms of two functions, k, k', which I had occasion to tabulate for the purpose of finding the resistance of a viscous fluid to a pendulum of the form of a cylindrical rod.

Putting, as in my former paper,

$$f_1(r) = f_0'(r), \quad \ldots \quad (13)$$

(f_1, f_0, are the functions there denoted by F_2, F_3), we have

$$f_0''(r) + \frac{1}{r} f_0'(r) - m^2 f_0(r) = 0. \quad \ldots \quad (14)$$

Now in both problems (that of my former paper and that of the present note) the function $f_0(r)$ satisfies the same differential equation (14) and the same condition of vanishing at infinity. Hence the function $f_0(r)$ is the same in the two cases, save as to the value of the arbitrary constant, which is a factor of the whole, and which disappears from the expression (12) as well as from those of k and k'.

The definition of k and k' is given by equation (99) of my former paper, viz. :—

$$1-\frac{4f_0'(a)}{m^2af_0(a)}=k-ik'. \quad \ldots \ldots \ldots \quad (15)$$

Now, by (13), (14), and (15),

$$1-\frac{af_1'(a)}{f_1(a)}=1-\frac{af_0''(a)}{f_0'(a)}=2-\frac{m^2af_0(a)}{f_0'(a)}=2\,\frac{k+1-ik'}{k-1-ik'}=2\,\frac{k^2-1+k'^2+2ik'}{(k-1)^2+k'^2};$$

whence we get, as before, for the part of the logarithmic decrement due to the external air, in consequence of the rotations of the two cylinders round their own axes, M′ denoting the mass of air which would be displaced by one if solid and of radius a,

$$\text{Nap. log. dec.}=\frac{4\text{M}'\mu'\tau}{\text{MK}^2}\cdot\frac{k^2-1+k'^2}{(k-1)^2+k'^2}. \quad \ldots \ldots \ldots \quad (16)$$

In the Table given in Art. 37 of my paper, $\mathfrak{m}$ denotes half the modulus of ma, or

$$\frac{a}{2}\sqrt{\frac{\pi}{\mu'\tau}}.$$

This Table is not available for calculating the effect of the internal air, for which we must have recourse to the differential equation (9). The integral of this equation, expressed in ascending series, subject to the condition of not becoming infinite at the origin, is

$$f_1(r)=\text{A}\left\{r+\frac{m^2r^3}{2.4}+\frac{m^4r^5}{2.4^2.6}+\frac{m^6r^7}{2.4^2.6^2.7}+\ldots\right\},$$

which gives

$$\frac{af_1'(a)}{f_1(a)}-1=\frac{\dfrac{m^2a^2}{4}+\dfrac{m^4a^4}{2.4.6}+\dfrac{m^6a^6}{2.4^2.6.8}+\ldots}{1+\dfrac{m^2a^2}{2.4}+\dfrac{m^4a^4}{2.4^2.6}+\dfrac{m^6a^6}{2.4^2.6^2.8}+\ldots}. \quad \ldots \ldots \quad (17)$$

Let the numerator of this fraction be denoted by $\text{E}+i\text{F}$, and the denominator by $\text{G}+i\text{H}$, where E, F, G, H, are real; then the real part will be $\text{EG}+\text{FH}$ divided by G^2+H^2, and we shall have for the correction due to the internal air

$$\text{Nap. log. dec.}=\frac{2\text{M}'\mu'}{\text{MK}^2}\cdot\frac{\text{EG}+\text{FH}}{\text{G}^2+\text{H}^2}. \quad \ldots \ldots \ldots \quad (18)$$

When the modulus of ma is small, it is rather more convenient to expand (17) according to ascending powers of ma. This may be done by actual division, or more conveniently by assuming a series with indeterminate coefficients, and using the non-linear differential equation of the first order in z obtained from (9) by putting $f_1'(r)=zf_1(r)$. Carried as far as to a^{12}, the development is

$$\frac{m^2a^2}{4}-\frac{m^4a^4}{96}+\frac{m^6a^6}{1536}-\frac{m^8a^8}{23040}+\frac{13m^{10}a^{10}}{4423680}-\frac{11m^{12}a^{12}}{55050240}:$$

and, denoting the modulus of ma by f, and taking the real part, we have

$$\text{Nap. log. dec.} = \frac{2M'\mu'}{MK^2}\left\{\frac{f^4}{96}-\frac{f^8}{23040}+\frac{11f^{12}}{55050240}-\ldots\right\} \quad . \quad . \quad . \quad . \quad (19)$$

This series must not be used when f is at all large, as the convergence is too slow, and, as appears by a theorem due to CAUCHY, it becomes actually divergent when $f=3\cdot340$* nearly, whereas the series in (17) are always convergent, and when f has the above value converge rapidly.

When f is decidedly large the series in (17), though ultimately convergent, begin by diverging, so that the calculation is troublesome, and moreover my Table giving k and k' is not carried beyond $f=8$, as the calculation by a different method then becomes very easy. In this case we should employ the integral of (9), which is of the form e^{-mr} or e^{mr} multiplied by a descending series. The former exponential only will come in when we are treating of the external air, and the latter only when of the internal.

For the external air the integral is of the form

$$f_1(r) = Be^{-mr}r^{-\frac{1}{2}}\left\{1+\frac{1.3}{1.(8mr)}-\frac{1^2.3.5}{1.2(8mr)^2}+\frac{1^2.3^2.5.7}{1.2.3(8mr)^3}-\ldots\right\}, \quad . \quad . \quad (20)$$

the signs being alternately + and −, and the new factors in the numerator being two less and two greater than the last factor in the term before. We get from (12), (20), and the expression for the logarithmic decrement in terms of T and the moment of inertia,

* The square root of the smallest real root of the equation

$$1-\frac{x}{2.4}+\frac{x^2}{2.4^2.6}-\ldots = 0.$$

The series would have become divergent still earlier if the equation just written had had an imaginary root with a modulus smaller than $3\cdot340\ldots^2$.

$$\text{Nap. log. dec.} = \frac{2M'\mu'\tau}{MK^2} \times \text{ real part of } ma \frac{1+\frac{3.5}{1.8ma}+\frac{1.3.5.7}{1.2(8ma)^2}-\frac{1^2.3.5.7.9}{1.2.3(8ma)^3}+\ldots}{1+\frac{1.3}{1.8ma}-\frac{1^2.3.5}{1.2(8ma)^2}+\frac{1^2.3^2.5.7}{1.2.3(8ma)^3}-\ldots} \quad . \quad (21)$$

Instead of the latter part of (21), in which, however, the law of either series is manifest, we may use its development according to descending powers of a, which is

$$ma+\frac{3}{2}+\frac{3}{8ma}-\frac{24}{(8ma)^2}+\frac{252}{(8ma)^3}-\frac{3456}{(8ma)^4}+\frac{60768}{(8ma)^5}-\frac{1327104}{(8ma)^6}+\ldots \quad . \quad . \quad (22)$$

The expression for the correction for the internal air will be got from the above by changing the sign of ma and of the whole, or, in other words, by changing the signs of the 2nd, 4th, 6th . . . terms in the series in (21) or (22). It will be remembered that ma is $f(\cos 45°+i \sin 45°)$.

APPENDIX.

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In the previous experiments the *main* loss of energy arising from the friction of the air may be characterised as being due to the fact that the air is *pushed*. A small portion, however, of the loss is occasioned by the rotation of the cylinders or spheres about their own axes, and in this case the air may be said to be *dragged*. Professor G. G. STOKES has, in the preceding note, deduced formulæ by means of which this last portion of the whole loss of energy can be calculated, and it seemed of interest to determine whether the coefficient of viscosity of air would prove to be the same as before, when the air was *entirely dragged*. This will occur when only one sphere or one cylinder is used, whose axis is made to coincide with the axis of rotation. Accordingly I followed out a suggestion of Professor STOKES in the manner detailed in the following experiments.

Experiment VI.

A paper cylinder was made by wrapping drawing-paper several times round a metal cylinder, which had been turned true throughout its whole length, the different layers being pasted together. When dry, the paper cylinder was removed from its metal core, and its external diameter very carefully gauged by calipers reading to $\frac{1}{1000}$th of an inch at six different places equidistant from each other. It was then gauged at the same distances from the ends, but in directions at right angles to the first. The following were the two sets of gauges :—

	Set I. Diameter in inches.	Set II. Diameter in inches.
	6·026	6·073
	6·083	6·010
	6·106	6·051
	6·106	6·020
	6·090	6·030
	6·010	6·006
Mean	6·0701	6·0323

The circumference of the cylinder was next measured by a steel tape at five different equidistant places :—

	Circumference in centims.
	48·64
	48·66
	48·60
	48·56
	48·35
Mean	48·562

Allowing for the thickness of the steel tape, the circumference is 48·485.

From the measurements made with the calipers and tape, the mean diameter of the cylinder was 15·370 and 15·433 centims. respectively, and the total mean 15·4015 centims.

It will be observed that the external diameter is nearly, but not quite, uniform throughout; this no doubt arises from the fact that the paper was not quite uniform in thickness. Inside, as far as could be judged by inserting a straight edge, the bore of the cylinder was perfectly uniform throughout.

The inside diameter was determined by the calipers at the top and bottom, at eight different places in all. It was also determined by gauging the thickness of the walls of the cylinder at the top and bottom by means of a wire gauge, and subtracting twice the thickness from the external diameter as measured by the tape. The internal diameter, measured in the two different ways mentioned above, was exactly the same for both, namely, 14·872 centims. The mean of the internal and external diameters is 15·1395 centims., and the mean radius 7·5698 centims.

The length of the paper cylinder was 60·80 centims., and the mass, allowing for the air displaced, was 543·6 grammes.

The wire was inserted into a hole bored in the centre of one end of a vertical brass rod 2 millims. thick and 15 centims. long, and there soldered: the other extremity of the rod was soldered into the centre of a horizontal, hollow, brass tube, of length 17·85 centims., of diameter 1·25 centim., and of mass 29·20 grammes.

From the hollow brass bar the paper cylinder was suspended; two holes, whose centres were $2\frac{1}{2}$ centims. from the top, being cut in the walls of the paper cylinder for this purpose.

Great care was taken in arranging the cylinder, so that the axis of rotation might coincide in direction as accurately as possible with the axis of the suspended system. As the paper cylinder did not quite hang truly, it was made to do so by placing small strips of tinfoil, as riders, on the top of the cylinder, and these strips were carefully padded down by hand to the walls of the cylinder. The usual previous precautions having been taken, the logarithmic decrement was determined from seven sets of observations, each involving 100 vibrations, as follows :—

Number of observation.	Logarithmic decrement.
1	·0026307
2	·0025856
3	·0025837
4	·0025810
5	·0025700
6	·0025849
7	·0025550

These observations were consecutive, and the mean of them is ·0025844.

The paper cylinder was now removed, and in its place was substituted a much shorter cylinder, made partly of paper and partly of tinfoil, and having nearly the same mass and mean radius. The dimensions of this cylinder were as carefully measured as those of the longer cylinder, with both steel tape and calipers. The mean of the inside and outside radius was 7·5132 centims., and its real length was 12·80 centims. Since the radius is, however, not quite the same as that of the longer cylinder, we must assume its length to be

$$12{\cdot}80 \times \left(\frac{7{\cdot}5132}{7{\cdot}5698}\right)^3 \text{ centims.,}$$

or 12·52 centims. if we are to use it for the purpose mentioned below.

The same pieces of tinfoil as had been used with the long cylinder were used here, and for the same purpose. The logarithmic decrement was then determined by six sets of experiments, each involving three times the number of vibrations employed with the longer cylinder.

Number of observation.	Logarithmic decrement.
1	·0009162
2	·0009015
3	·0008743
4	·0008871
5	·0009019
6	·0008993

These, like the others, are consecutive observations, and the mean of them is ·0008967.

Applying the corrections, mentioned in the paper, for small differences in the vibration-periods, temperature, &c., when the two cylinders were used, we have for the logarithmic decrement due to a cylinder (60·80 — 12·32) centims. or 48·28 centims. in length the value

$$\cdot 0017029.$$

It follows, from Professor STOKES's formulæ, that the logarithmic decrement arising from the friction of the air against the inner and outer walls taken together will be

$$\frac{M\mu\tau}{I\rho}\log_{10} e\left(\sqrt{2}f+\sqrt{2}\times 0\cdot 375f^{-1}-\sqrt{2}\times 0\cdot 4922f^{-3}+\&c.\right),$$

$$f \text{ being equal to } \sqrt{\frac{\pi\rho}{\mu\tau}}.a,$$

where a is the mean radius of the cylinder, τ the vibration-period, μ the coefficient of viscosity, ρ the density of the air, M the mass of air which would be contained in a cylinder of the same length, and having an internal radius equal to a, and I the moment of inertia.

The values of I and τ were 36966 centimetre-grammes and 3·6038 seconds respectively. The corrected height of the barometer was 29·354 inches, and the temperature 12°·225 C. The value of ρ was calculated, as usual, on the supposition that the air is half saturated with moisture.

The terms $0\cdot 375f^{-1}$ and $0\cdot 4922f^{-3}$ are so small that we may calculate them by using an approximate value of μ, and the series converges so rapidly that it is quite unnecessary to include any more terms in it.*

The value of μ, determined from the data given above, was found to be

$$\cdot 00017580.$$

* Indeed, the third term might have been dispensed with in this case, but not in the next experiment.

Experiment VII.

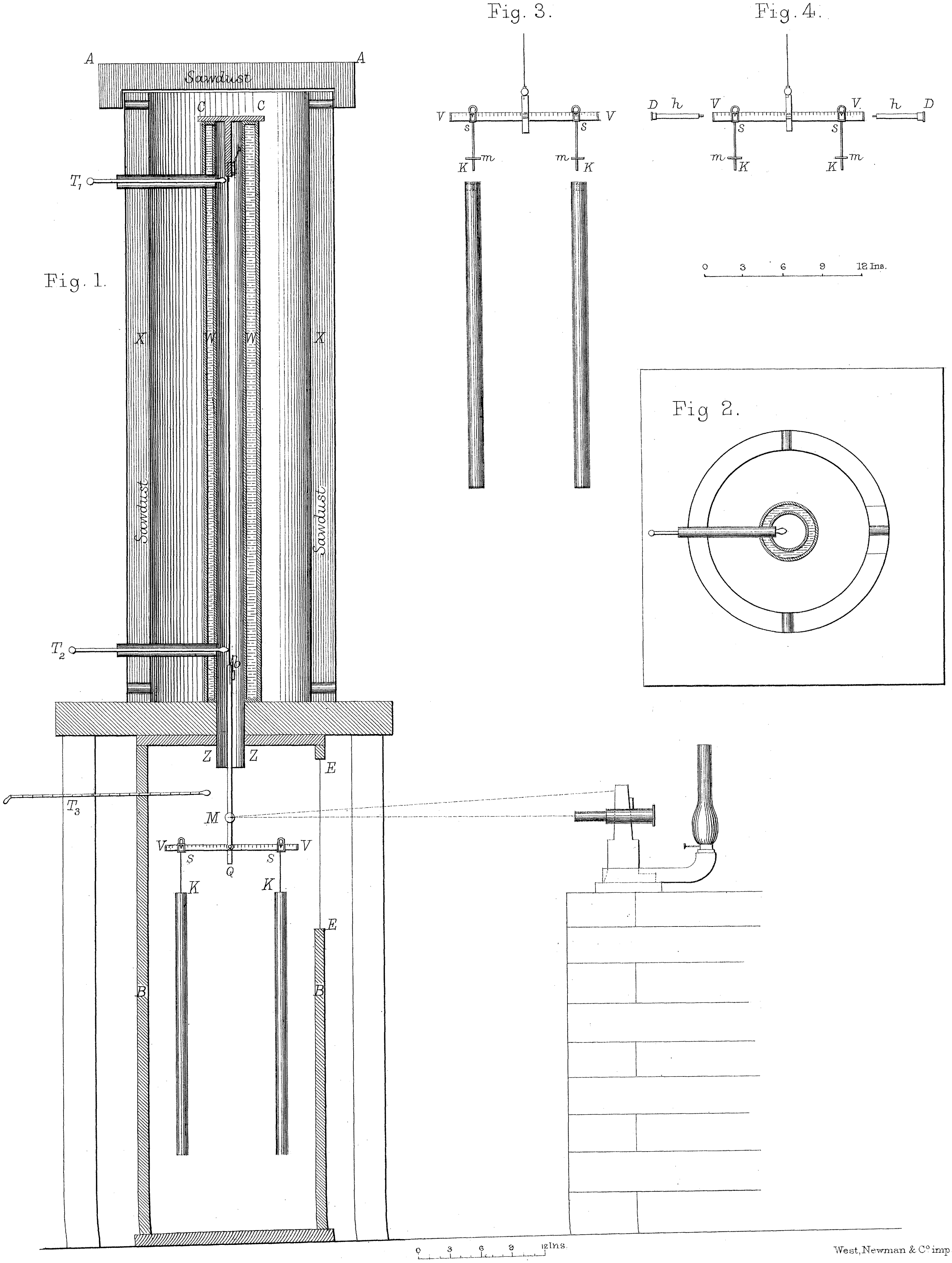
The copper wire used in the last experiment was about $4\frac{1}{2}$ feet in length and 0·1 centim. in diameter. This was now changed for one of the same length, but of 0·063 centim. diameter, so that the vibration-period became 8·930 seconds. The rest of the arrangements were the same as in Experiment VI. The corrected logarithmic decrement was ·0027040, and the value of μ deduced as above was found to be ·00017902 at a temperature of 13°·100 C.

The mean of the two last experiments is ·00017741 at a temperature of 12°·663 C. This result agrees so well with the mean of those deduced from the previous experiments that it is unnecessary to make any alteration in the formula already given for finding the viscosity at any temperature.

I have entered more into the details of these last experiments, as I think the present method can be more advantageously employed than any of the others. Indeed, by spending sufficient time over the experiments, whereby the errors likely to arise from the somewhat unstable nature of the internal friction of the metal may be more perfectly eliminated, it seems likely that very considerable accuracy can be attained by it.

[NOTE added Dec. 8th, 1886.—A much greater number of observations were afterwards made with the same cylinders and wires, and resulted as follows :—With the wire used in Experiment VI. the value of μ obtained was ·00017708 at a temperature of 12°·225 C., and with the finer wire of Experiment VII. the value was ·00017783 at a temperature of 13°·075 C. The mean of these values is ·00017746 at 12°·650 C., as compared with ·00017711 at 11°·79 C., the mean of the other five sets of experiments. If we allow for the difference of temperature by using the previously given formula, the agreement between these two means is perfect.]

In conclusion, my warmest thanks are due to Professor STOKES for his valuable suggestions and advice throughout the investigation. To myself the experimental verification of Professor STOKES's formulæ has been a source of great pleasure.



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